

THE COHOMOLOGY RING OF PRODUCT COMPLEXES⁽¹⁾

BY
F. P. PALERMO

Introduction. The Künneth formula enables one to determine the integral cohomology group of the product $X \times Y$ of the spaces X and Y in terms of the integral cohomology groups $H(X)$ and $H(Y)$. However, this formula does not enable one to determine the multiplicative structure of the cohomology ring $H(X \times Y)$ in terms of the integral cohomology rings⁽²⁾ $H(X)$ and $H(Y)$. It is natural to ask the question: Is the integral cohomology ring $H(X \times Y)$ determined by the integral cohomology rings $H(X)$ and $H(Y)$?

This question is answered in the negative by the following example:

Let $X_1 = Y_1$ be the union of the real projective plane and a one-sphere (circle) with one point in common. Let $X_2 = Y_2$ be a Klein bottle. It is easy to check that the rings $H(X_1)$ and $H(X_2)$ are isomorphic. However, the rings $H(X_1 \times Y_1)$ and $H(X_2 \times Y_2)$ are not isomorphic. In particular, there is a one-dimensional cohomology class and a three-dimensional cohomology class in $H(X_2 \times Y_2)$ whose product is a nonzero four-dimensional cohomology class. On the other hand, all products of one-dimensional and three-dimensional cohomology classes of $H(X_1 \times Y_1)$ are zero. Hence these two cohomology rings cannot be isomorphic. These assertions will follow readily from the theorems proved below.

Since the answer to the above question is negative, it is natural to inquire: What information about the cohomology rings of X and Y is needed to determine the integral cohomology ring of $X \times Y$? Let $H(X, n)$ denote the cohomology ring of X with the integers modulo n as coefficients. Following J. H. C. Whitehead [8], and Bockstein [2], we define the *spectrum of cohomology rings of X* , or simply the *cohomology spectrum of X* , to be the set of cohomology rings $H(X, n)$, $n \geq 0$, together with the coefficient homomorphisms: $H(X, n) \rightarrow H(X, m)$, $m > 0$ and the Bockstein homomorphisms of degree $+1$: $H(X, m) \rightarrow H(X, 0)$, $m > 0$. (These homomorphisms are defined in section (1).) Bockstein [2] stated but did not prove that the cohomology spectra of X and Y determine the cohomology ring $H(X \times Y)$. From these spectra he constructed a group isomorphic to the group $H(X \times Y)$, however, he did not introduce any products in this construction; thus the question of

Presented to the Society, under the title *The cohomology rings of product spaces* on December 27, 1954; received by the editors March 11, 1956 and, in revised form, July 19, 1956.

⁽¹⁾ A condensed version of a doctoral dissertation submitted to Brown University. The author wishes to thank Professor W. S. Massey for his advice and guidance in the writing of this thesis.

⁽²⁾ The multiplication in the cohomology ring is the "cup" product. The cohomology ring is a graded, skew-commutative, associative ring.

the determination of the integral cohomology ring of $X \times Y$ was left open. Bockstein also gave a construction for determining the groups $H(X \times Y, n)$, the coefficient homomorphisms and the Bockstein homomorphisms.

In this paper, the problem of determining the cohomology ring of the product $X \times Y$ is treated in a purely algebraic fashion. We start with a cochain complex K , and study the cohomology spectrum of K . Various relations which exist within this spectrum are indicated in the first chapter of this paper. The tensor product $K \otimes L$ of two cochain complexes K and L corresponds to the cartesian product $X \times Y$ of the topological spaces X and Y . Thus, the algebraic problem amounts to the determination of the cohomology ring $H(K \otimes L)$. In the second chapter, Bockstein's construction is used to construct a group $R_0(K, L)$, which is isomorphic to the group $H(K \otimes L)$, if the complexes K and L are finitely generated. Products are introduced in $R_0(K, L)$, giving a ring which is proved to be isomorphic to the cohomology ring $H(K \otimes L)$. It is seen that only a finite collection of rings of the spectra of K and L are actually needed in the determination of the ring $H(K \otimes L)$ when K and L are finitely generated. Thus, this construction lends itself to computation.

In Chapter III, it is shown that the cohomology ring $H(K \otimes L, n)$ is determined by the spectra of K and L . The coefficient and Bockstein homomorphisms for $K \otimes L$ are considered in Chapter IV, and the various constructions are shown to commute with direct limits. Thus the result can be extended to infinite complexes and Čech cohomology.

CHAPTER I. THE COHOMOLOGY SPECTRUM OF A COCHAIN COMPLEX

1. Definitions and conventions. A pair $(C(K), \delta)$ is called a *graded differential group* if $C(K)$ is the direct sum of subgroups $C^r(K)$, where $C^r(K) = 0$ for $r < 0$, and δ is a differential operator homogeneous of degree $+1$ (or -1) i.e. δ is an endomorphism of K with $\delta C^r(K) \subset C^{r+1}(K)$ and $\delta\delta = 0$. An automorphism ω of $C(K)$ is said to be an *involution* of $C(K)$ if $\omega\delta + \delta\omega = 0$ and $\omega\omega(x) = x$ for all $x \in C(K)$. Usually ω will be the involution defined by $\omega(x) = (-1)^p x$ for $x \in C^p(K)$.

A triple $(C(K), \delta, \omega)$ is called a *graded differential ring* if $(C(K), \delta)$ is a graded differential group, ω is an involution, $C(K)$ has an identity $1 \in C^0(K)$, $x \cdot y \in C^{p+q}(K)$ for $x \in C^p(K)$ and $y \in C^q(K)$, and $\delta(xy) = (\delta x)y + \omega(x)\delta(y)$.

$K = (C(K), \delta, \omega)$ will be called a *cochain complex* if $(C(K), \delta, \omega)$ is a graded differential ring with δ homogeneous of degree $+1$.

We let $H(K, n)$ denote the cohomology ring of the cochain complex $K \otimes Z_n = (C(K) \otimes Z_n, \delta \otimes 1, \omega \otimes 1)$, i.e. $H(K, n) = \text{kernel } (\delta \otimes 1) \text{ modulo image } \delta \otimes 1$ where 1 is the identity on Z_n , the integers modulo n . If $n = 0$, $H(K, 0)$, shortened to $H(K)$, denotes the integral cohomology ring of K , i.e. kernel δ modulo image δ . The involution ω of K induces an involution ω on each $H(K, n)$. We let $(x)_n$ denote the cohomology class in $H(K, n)$ of a cochain

x which is a cocycle modulo n , i.e. $\delta x = ny$ for some cochain y in K .

For each pair of integers $m \geq 0$, $n > 0$, the coefficient homomorphisms $h_{n,m}: H(K, m) \rightarrow H(K, n)$ are defined by the formula

$$h_{n,m}(x_m) = \left(\frac{n}{(n, m)} x \right)_n$$

for any cochain x which is a cocycle modulo m , where (m, n) denotes the greatest common divisor of m and n . It is easy to check that $h_{n,m}$ is well defined, and is a group homomorphism which is homogeneous of degree zero.

For $n > 0$, the Bockstein homomorphism $\Delta_n: H(K, n) \rightarrow H(K)$ is defined by the formula

$$\Delta_n(x_n) = \left(\frac{1}{n} \delta x \right)_0.$$

It is easy to check that Δ_n is a well defined group homomorphism which is homogeneous of degree $+1$.

2. The cohomology spectrum of a cochain complex. The *cohomology spectrum of a cochain complex* is defined to be the collection of cohomology rings $H(K, n)$ together with the Bockstein and coefficient homomorphisms.

We shall now list some relations which exist between the Bockstein and coefficient homomorphisms of the cohomology spectrum of the cochain complex K . Their proofs are consequences of the definitions. Most of these formulae, together with their proofs, are in J. H. C. Whitehead [8].

The following two conditions are satisfied by the Bockstein and coefficient homomorphisms:

$$(2.1) \quad h_{k,m} h_{m,n} = \frac{m(n, k)}{(m, k)(m, n)} h_{k,n},$$

$$(2.2) \quad \Delta_m h_{m,n} = \frac{n}{(m, n)} \Delta_n.$$

Let $\sum(K)$ denote the direct sum of the rings $H(K, n)$ for all $n \geq 0$. A product is introduced in $\sum(K)$ by the formula

$$(2.3) \quad xy = h_{c,m}(x) h_{c,n}(y)$$

where x is in $H(K, m)$, y is in $H(K, n)$ and $c = (m, n)$. This product is extended to all of $\sum(K)$ by linearity. It is easy to check that $\sum(K)$ is an associative skew commutative ring with this product.

The Bockstein and coefficient homomorphisms satisfy the following relations with respect to these products:

$$(2.4) \quad h_{k,m}(x) h_{k,n}(y) = \frac{k(k, m, n)}{(k, m)(k, n)} h_{k,c}(xy),$$

$$(2.5) \quad \Delta_{c,c}(xy) = [\Delta_{c,c}h_{c,m}(x)]y + \omega(x)[\Delta_{c,c}h_{c,n}(y)],$$

where $x \in H(K, m)$, $y \in H(K, n)$, $c = (m, n)$ and $\Delta_{c,c} = h_{c,0}\Delta_c$. If $k \mid n$ and $m = n$, formula (2.4) reduces to

$$(2.6) \quad h_{k,m}(x)h_{k,m}(y) = h_{k,m}(xy)$$

i.e. $h_{k,m}: H(K, m) \rightarrow H(K, k)$ is a ring homomorphism if k divides m .

The following two relations have some important consequences:

$$(2.7) \quad \Delta_n(y \cdot h_{n,0}(x)) = \Delta_n(y) \cdot x,$$

$$(2.8) \quad \Delta_n(h_{n,0}(x) \cdot y) = \omega(x) \cdot \Delta_n(y)$$

for $x \in H(K)$ and $y \in H(K, n)$. If in formulae (2.7) and (2.8) $x = \Delta_n(z)$ for some z in $H(K, n)$, these equations take the form

$$\Delta_n(y \cdot \Delta_{n,n}(z)) = \Delta_n(y) \cdot \Delta_n(z),$$

$$\Delta_n(\Delta_{n,n}(z) \cdot y) = \omega(\Delta_n(z)) \cdot \Delta_n(y).$$

These last two formulae suggest that $\Delta_n: H(K, n) \rightarrow H(K)$ will become a ring homomorphism if a new product $*$ is introduced in $H(K, n)$ by either one of the formulae

$$x * y = x \cdot \Delta_{n,n}(y),$$

$$x * y = \omega \Delta_{n,n}(x) \cdot y$$

for x and y in $H(K, n)$.

Let K and K' be cochain complexes and suppose $F: K \rightarrow K'$ is an allowable cochain map, i.e., F satisfies the following conditions:

$$(i) \quad F\delta = \delta F,$$

$$(ii) \quad F\omega = \omega F.$$

Then F induces homomorphisms $F_n: H(K, n) \rightarrow H(K', n)$ for each n . These induced homomorphisms are related to the Bockstein and coefficient homomorphisms and the multiplication by the formulae

$$(a) \quad F_n h_{n,m} = h_{n,m} F_m,$$

$$(2.9) \quad (b) \quad F_0 \Delta_n = \Delta_n F_n,$$

$$(c) \quad F_n(xy) = F_n(x)F_n(y),$$

where x and y are in $H(K, n)$.

The Bockstein and coefficient homomorphisms preserve direct sums. Stated precisely, we have the direct sum theorem:

THEOREM 2.1. *If $K = \sum_i K_i$ (direct sum) where each K_i is an allowable subcomplex (i.e. $\delta K_i \subset K_i$) of the cochain complex K , then*

$$(a) \quad H(K, n) = \sum_i H(K_i, n) \text{ for each } n. \text{ (Direct sum.)}$$

$$(b) \quad h_{m,n}(H(K_i, n)) \subset H(K_i, m).$$

$$(c) \quad \Delta_n(H(K_i, n)) \subset H(K_i).$$

CHAPTER II. THE INTEGRAL COHOMOLOGY RING $H(K \otimes L)$

3. The group $H(K \otimes L)$. Let K and L be cochain complexes. The tensor product $K \otimes L$ of the cochain complexes K and L is defined to be the graded differential ring $(C(K) \otimes C(L), (\delta \otimes 1) + (\omega \otimes \delta), \omega \otimes \omega)$ with multiplication defined by the formula

$$(3.1) \quad (x \otimes y) \cdot (u \otimes v) = (-1)^{pq} xu \otimes yv$$

for $y \in C^p(K)$ and $u \in C^q(L)$. Let $S_0(K, L)$ be the weak direct sum of the groups $H(K, n) \otimes H(L, n)$ for all non-negative integers n . Define $\phi_0: S_0(K, L) \rightarrow H(K \otimes L)$ as follows:

$$\begin{aligned} \phi_0|H(K) \otimes H(L) &= \alpha_0, \\ \phi_0|H(K, n) \otimes H(L, n) &= \Delta_n \alpha_n \end{aligned}$$

where $\alpha_n: H(K, n) \otimes H(L, n) \rightarrow H(K \otimes L, n)$ is the homomorphism induced by the bilinear map of $H(K, n) \times H(L, n) \rightarrow H(K \otimes L, n)$ which carries (x_n, y_n) into $(x \otimes y)_n$. Thus we have $\alpha_n(x_n \otimes y_n) = (x \otimes y)_n$.

THEOREM 3.1. *If K and L are finitely generated free cochain complexes, then*

(a) ϕ_0 is a homomorphism onto,

(b) kernel ϕ_0 is the subgroup $Q_0(K, L)$ of $S_0(K, L)$

generated by the following relators:

- | | |
|---|------------------------|
| (i) $h_{i,j}(x_j) \otimes y_i - x_j \otimes h_{j,i}(y_i),$ | $i j \text{ or } j i,$ |
| (ii) $\Delta_i(x_i) \otimes y_0 - x_i \otimes h_{i,0}(y_0),$ | $i > 0,$ |
| (iii) $x_0 \otimes \Delta_i(y_i) - h_{i,0}(\omega(x_0)) \otimes y_i,$ | $i > 0.$ |

Proof. A finitely generated free cochain complex is the direct sum of elementary allowable subcomplexes, an elementary complex being a complex E of one of the following two types:

(1) E has one free generator c with $\delta c = 0$,

(2) E has two free generators a and b with $\delta a = tb$ and $\delta b = 0$ where t is an integer greater than 1.

Since K and L are finitely generated free cochain complexes, they each split into the direct sum of elementary complexes. Hence it suffices to prove (a) and (b) for the case where K and L are elementary complexes⁽³⁾ because of the following direct sum lemma.

LEMMA 3.2. *If $K = \sum_i K_i$, $L = \sum_j L_j$ where K_i and L_j are allowable subcomplexes of K and L respectively, then*

⁽³⁾ Some details of such a proof are given in the appendix.

$$(i) S_0(K, L) = \sum_{i,j} S_0(K_i, L_j),$$

$$(ii) \phi_0(S_0(K_i, L_j)) \subset H(K_i \otimes L_j).$$

The proof of this lemma is a straightforward verification using the facts that homology and tensor products preserve direct sums (Eilenberg and Steenrod [5, p. 141]).

4. The multiplication in $H(K \otimes L)$. Thus far, the ring structure of $H(K \otimes L)$ has not been considered. The multiplication in $K \otimes L$ is given by formula (3.1). In this section products will be introduced in $S_0(K, L)$. It is convenient to write $h'_{i,j}$ in place of $h_{i,j} \otimes h_{i,j}$: $H(K, j) \otimes H(L, j) \rightarrow H(K, i) \otimes H(L, i)$.

Since $H(K, n)$ and $H(L, n)$ are graded rings, $H(K, n) \otimes H(L, n)$ is given a ring structure in the standard way, i.e. for x in $H^p(K, n)$, y in $H^q(L, n)$, u in $H(K, n)$ and v in $H(L, n)$

$$(4.1) \quad (u \otimes y) \cdot (x \otimes v) = (-1)^{pq}(ux) \otimes (yv).$$

With this multiplication, $H(K, n) \otimes H(L, n)$ is an associative ring. This multiplication is extended to $S_0(K, L) = \sum_{n \geq 0} H(K, n) \otimes H(L, n)$ by the formula

$$(4.2) \quad x \cdot y = h'_{c,i}(x) \cdot h'_{c,j}(y)$$

where x is in $H(K, i) \otimes H(L, i)$, y is in $H(K, j) \otimes H(L, j)$ and $c = (i, j)$. With this multiplication, $S_0(K, L)$ is an associative ring.

In §2, a multiplication was introduced in the direct sum of the cohomology rings $H(K, i)$, $i \geq 0$. We list some properties of the homomorphisms α_i : $H(K, j) \otimes H(L, j) \rightarrow H(K \otimes L, j)$ and the coefficient homomorphisms with respect to these products.

$$(4.3) \quad \alpha_i h'_{i,j} = \frac{i}{(i,j)} h_{i,j} \alpha_j, \quad i > 0, j \geq 0,$$

$$(4.4) \quad \alpha_i(xy) = \alpha_i(x)\alpha_i(y) \text{ for } x \text{ and } y \text{ in } H(K, i) \otimes H(L, i).$$

$$(4.5) \quad \alpha_i h'_{i,j} = h_{i,j} \alpha_j \text{ if } i \mid j,$$

$$(4.6) \quad \alpha_c(xy) = \alpha_i(x) \cdot \alpha_j(y) \text{ for } x \text{ in } H(K, i) \otimes H(L, i), \\ y \text{ in } H(K, j) \otimes H(L, j) \text{ and } c = (i, j).$$

Formulae (4.3) and (4.4) follow immediately from the definitions while formulae (4.5) and (4.6) are consequences of (4.3) and (4.4).

However, $\phi_0: S_0(K, L) \rightarrow H(K \otimes L)$ does not preserve these products. This can be seen just from considerations of degrees. We noted in §2 that a new multiplication $*$ can be introduced in $H(K, n)$ so that $\Delta_n: H(K, n) \rightarrow H(K)$ becomes a ring homomorphism. Following this lead, we introduce a coboundary endomorphism D_n in $H(K, n) \otimes H(L, n)$ for $n > 0$ by the formula

$$(4.7) \quad D_n = \Delta_{n,n} \otimes I + \omega \otimes \Delta_{n,n}$$

where I is the identity on $H(L, n)$ and ω is the involution of $H(K, n)$. An involution ω' of $H(K, n) \otimes H(L, n)$ is defined by $\omega' = \omega \otimes \omega$.

The following properties of D_n and ω' are easily verified using formulae (4.3), (4.4), (4.5) and (4.6).

$$(4.8) \quad D_n D_n = 0, \quad D_n \omega' + \omega' D_n = 0.$$

$$(4.9) \quad \alpha_n D_n = \Delta_{n,n} \alpha_n, \quad \alpha_n \omega' = \omega \alpha_n.$$

$$(4.10) \quad \text{If } c \mid i, \quad D_c h'_{c,i} = \frac{i}{c} h'_{c,i} D_i.$$

$$(4.11) \quad \Delta_n \alpha_n (x \cdot D_n(y)) = \Delta_n \alpha_n (x) \Delta_n \alpha_n (y) = \Delta_n \alpha_n (\omega' D_n(x) \cdot y),$$

where x and y are in $H(K, n) \otimes H(L, n)$.

A multiplication $*$ is defined in $S_0(K, L)$ as follows: For each pair of positive integers i and j , choose integers α and β so that $\alpha i + \beta j = (i, j)$. For x in $H(K, i) \otimes H(L, i)$, y in $H(K, j) \otimes H(L, j)$, we define

$$(4.12) \quad \begin{aligned} x * y &= xy \text{ if } j = 0 \\ &= \omega'(x)y \text{ if } j > 0, i = 0 \\ &= \alpha x \cdot D_j(y) + \beta \omega'(D_i(x)) \cdot y \text{ if } i > 0, j > 0. \end{aligned}$$

This product is extended to $S_0(K, L)$ by linearity. $S_0(K, L)$ with this multiplication is a (nonassociative) ring.

The following theorem shows the importance of the $*$ products.

THEOREM 4.1. $\phi_0: S_0(K, L) \rightarrow H(K \otimes L)$ is a ring homomorphism with respect to the $*$ products in $S_0(K, L)$; i.e. $\phi_0(x * y) = \phi_0(x) \cdot \phi_0(y)$.

Proof. It suffices to check this for

$$x \text{ in } H(K, i) \otimes H(L, i) \text{ and } y \text{ in } H(K, j) \otimes H(L, j).$$

There are four cases to be considered. Case 1, $i=j=0$ follows from formula (4.4); Case 2, $i>0, j=0$; and Case 3, $i=0, j>0$ follow from formulae (4.6), (2.6) and (2.8). Case 4, $i>0, j>0$ follows from formulae (4.13) and (4.14), below, which are consequences of formulae (4.9), (4.10) and (4.11).

For x in $H(K, i) \otimes H(L, i)$ and y in $H(K, j) \otimes H(L, j)$, $i>0, j>0, c=(i, j)$,

$$(4.13) \quad \Delta_c \alpha_c (x D_i(y)) = \frac{i}{c} \Delta_i \alpha_i (x) \Delta_j \alpha_j (y),$$

$$(4.14) \quad \Delta_c \alpha_c (\omega'(D_i(x)) y) = \frac{j}{c} \Delta_i \alpha_i (x) \Delta_j \alpha_j (y).$$

We should observe that in this theorem we have not assumed K and L to be finitely generated cochain complexes.

Combining Theorems 3.1 and 4.1, we obtain

COROLLARY 4.2. *If K and L are finitely generated free cochain complexes, then $Q_0(K, L)$ is an ideal of $S_0(K, L)$ with $*$ as multiplication.*

Thus we have the very important

COROLLARY 4.3. *ϕ_0 induces a ring isomorphism ϕ_0^* of $S_0(K, L)$ modulo $Q_0(K, L)$ onto $H(K \otimes L)$ when K and L are finitely generated and free.*

The multiplication $*$ introduced in $S_0(K, L)$ depended on choices of integers α and β . Theorem 4.1 shows that this multiplication is determined modulo kernel ϕ_0 . Thus if K and L are finitely generated and free, $*$ is determined modulo $Q_0(K, L)$. However, it can be shown directly that if a second multiplication $*'$ is introduced in $S_0(K, L)$ using a different choice of integers, then for x in $H(K, i) \otimes H(L, i)$, y in $H(K, j) \otimes H(L, j)$ and $c = (i, j)$

$$x * y - x *' y = D_c(k\omega'(x)y) \text{ for some integer } k.$$

But $D_c(u)$ is easily seen to be in $Q_0(K, L)$. Thus $*$ is unique modulo $Q_0(K, L)$.

Suppose K and L are finitely generated free cochain complexes. Let T be the collection of all integers n such that $n=0$ or $n=(t, s)$ where t ranges over the torsion coefficients of K and s ranges over the torsion coefficients of L (Eilenberg and Steenrod [5, p. 138]). If we examine the proof that ϕ_0 is onto $H(K \otimes L)$, we see that only the groups $H(K, n) \otimes H(L, n)$ for n in T are needed. However $\sum_{n \in T} H(K, n) \otimes H(L, n)$ need not be a subring of $S_0(K, L)$. Let T' be the smallest collection of integers which contains T and is closed with respect to taking greatest common divisors. Then clearly

$$\sum_{n \in T'} H(K, n) \otimes H(L, n).$$

is a subring of $S_0(K, L)$ which contains the subring generated by

$$\sum_{n \in T} H(K, n) \otimes H(L, n).$$

Thus the cohomology ring $H(K \otimes L)$ is determined by the finite collection of cohomology rings $H(K, n)$ and $H(L, n)$ for n in T' if K and L are finitely generated and free.

CHAPTER III. THE COHOMOLOGY RING $H(K \otimes L, n)$

5. **The group $H(K \otimes L, n)$, $n > 0$.** Suppose K and L are cochain complexes. Let $P_n(K, L)$ be the direct sum of the groups $H(K, i) \otimes H(L, i)$ for all integers i which divide n . Let $S_n(K, L)$ be the group direct⁽⁴⁾ product

$$P_n(K, L) \times P_n(K, L).$$

We define $\theta_n: P_n(K, L) \rightarrow H(K \otimes L, n)$ by the formula

$$\theta_n | H(K, i) \otimes H(L, i) = h_{n,i} \alpha_i$$

⁽⁴⁾ We use the direct product notation here because both factors are the same, while for $P_n(K, L)$ we used the direct sum notation because all the factors were distinct.

and $\phi_n: P_n(K, L) \times P_n(K, L) \rightarrow H(K \otimes L, n)$ by the formula

$$(5.1) \quad \phi_n(x, y) = \theta_n(x) + \Delta_{n,n}\theta_n(y),$$

for x and y in $P_n(K, L)$. Clearly, ϕ_n is a group homomorphism.

THEOREM 5.1. *If K and L are finitely generated free cochain complexes, then*

(a) ϕ_n is a homomorphism onto $H(K \otimes L, n)$.

(b) kernel ϕ_n is the subgroup $Q_n(K, L)$ of $S_n(K, L)$ generated by the following relators:

$$(i) \quad (0, h_{i,j}(x_j) \otimes y_i - x_j \otimes h_{j,i}(y_i)),$$

$$(ii) \quad (h_{i,j}(x_j) \otimes y_i - x_j \otimes h_{j,i}(y_i), 0),$$

$$(iii) \quad (D_n(u), -u),$$

where i and j both divide n , $x_j \in H(K, j)$, $y_i \in H(L, i)$ and $u \in H(K, n) \otimes H(L, n)$.

As in Chapter II, it suffices to prove the theorem for K and L as elementary complexes⁽⁵⁾ because of the following lemma.

LEMMA 5.2. *If $K = \sum_i K_i$, $L = \sum_j L_j$ where K_i and L_j are allowable sub-complexes of the cochain complexes K and L respectively, then*

$$(1) \quad S_n(K, L) = \sum_{i,i} S_n(K_i, L_i),$$

$$(2) \quad \phi_n(S_n(K_i, L_j)) \subset H(K_i \otimes L_j, n).$$

The proof of this lemma is left to the reader.

Letting $R_n(K, L) = S_n(K, L) \bmod Q_n(K, L)$, we have

COROLLARY 5.3. *If K and L are finitely generated free cochain complexes, $\phi_n: S_n(K, L) \rightarrow H(K \otimes L, n)$ induces an isomorphism $\phi_n^*: R_n(K, L) \rightarrow H(K \otimes L, n)$, $n > 0$.*

6. Products in $S_n(K, L)$. Thus far $S_n(K, L)$ has been given only a group structure as the direct product $P_n(K, L) \times P_n(K, L)$.

A multiplication will now be introduced in $S_n(K, L)$ by first introducing a multiplication in $P_n(K, L) = \sum_{i|n} H(K, i) \otimes H(L, i)$. It is easy to see that $P_n(K, L)$ is an ideal of $S_0(K, L)$ with respect to the skew multiplication defined by formulae (4.1) and (4.2), and also with respect to the $*$ product defined by formula (4.12). Thus, in particular, $P_n(K, L)$ is closed with respect to both these multiplications. We introduce another product $\odot$ in $P_n(K, L)$ by defining for x in $H(K, i) \otimes H(L, i)$ and y in $H(K, j) \otimes H(L, j)$

$$(6.1) \quad x \odot y = \frac{n}{[i, j]} xy$$

⁽⁵⁾ Details of such a proof are given in the appendix.

where $[i, j]$ is the least common multiple of i and j . This product is extended to all of $P_n(K, L)$ by linearity.

The following formulae show the relations between the homomorphism θ_n and the products defined on $P_n(K, L)$. These formulae are established using formulae (2.4), (2.5) and the definition of θ_n .

$$(6.2) \quad \theta_n(x \odot y) = \theta_n(x)\theta_n(y);$$

$$(6.3) \quad \theta_n(x \cdot D_i(y)) = \frac{i}{c} \theta_n(x) \cdot \Delta_{n,n}\theta_n(y);$$

$$(6.4) \quad \theta_n(D_i(x) \cdot y) = \frac{j}{c} [\Delta_{n,n}\theta_n(x) \cdot \theta_n(y)];$$

$$(6.5) \quad \theta_n(x * y) = \alpha \frac{i}{c} \theta_n(x) \cdot \Delta_{n,n}(\theta_n(y)) + \beta \frac{j}{c} \omega(\Delta_{n,n}\theta_n(x)) \cdot \theta_n(y);$$

$$(6.6) \quad \Delta_{n,n}\theta_n = h_{n,0}\phi_0|P_n(K, L);$$

where $x \in H(K, i) \otimes H(L, i)$, $y \in H(K, j) \otimes H(L, j)$, $c = (i, j)$, α and β are integers such that $\alpha i + \beta j = c$, and $*$ is defined using these integers (formula (4.12)).

For each i and j which divides n , choose integers α and β such that $\alpha i + \beta j = c = (i, j)$ and suppose that $*$ has been defined using these integers. Let $x \in H(K, i) \otimes H(L, i)$, $y \in H(K, j) \otimes H(L, j)$. A multiplication denoted by $**$ is introduced in $S_n(K, L) = P_n(K, L) \times P_n(K, L)$ by the formulae

$$(6.7) \quad \begin{aligned} (i) \quad & (0, x) ** (0, y) = (0, x * y), \\ (ii) \quad & (0, x) ** (y, 0) = \left(-\omega'(x) * y, \alpha \frac{i}{c} x \odot y \right), \\ (iii) \quad & (y, 0) ** (0, x) = \left(y * x, \alpha \frac{i}{c} \omega'(y) \odot x \right), \\ (iv) \quad & (y, 0) ** (x, 0) = (y \odot x, 0). \end{aligned}$$

This product is extended to all of $S_n(K, L)$ by linearity. It is easy to check that $S_n(K, L)$ with this multiplication is a (nonassociative) ring.

THEOREM 6.1. $\phi_n: S_n(K, L) \rightarrow H(K \otimes L, n)$ is a ring homomorphism with respect to the $**$ product defined by formula (6.7), i.e. $\phi_n(u ** v) = \phi_n(u) \cdot \phi_n(v)$ for u and v in $S_n(K, L)$.

Proof. It suffices to prove this for the four cases given in formula (6.7). Cases (i) and (iv) are immediate consequences of formulae (6.2) and (6.6). The proofs for cases (ii) and (iii) are similar, hence we only give a detailed proof for Case (ii). Note the use of formulae (5.1), (6.5), (6.2) and (2.5).

$$\begin{aligned}
\phi_n((0, x) ** (y, 0)) &= \phi_n\left(-\omega'(x) * y, \alpha \frac{i}{c} x \odot y\right) \\
&= -\theta_n(\omega'(x) * y) + \alpha \frac{i}{c} \Delta_{n,n} \theta_n(x \odot y) \\
&= -\left[\alpha \frac{i}{c} \theta_n(\omega'(x)) \cdot \Delta_{n,n} \theta_n(y) \right. \\
&\quad \left. + \beta \frac{j}{c} \omega(\Delta_{n,n} \theta_n(\omega'(x))) \cdot \theta_n(y) \right] \\
&\quad + \alpha \frac{i}{c} [\Delta_{n,n}(\theta_n(x)) \cdot \theta_n(y) + \omega \theta_n(x) \cdot \Delta_{n,n} \theta_n(y)] \\
&= \beta \frac{j}{c} \Delta_{n,n} \theta_n(x) \cdot \theta_n(y) + \alpha \frac{i}{c} \Delta_{n,n} \theta_n(x) \cdot \theta_n(y) \\
&= \Delta_{n,n} \theta_n(x) \cdot \theta_n(y) \\
&= \phi_n(0, x) \cdot \phi_n(y, 0).
\end{aligned}$$

The following corollaries are immediate consequences of this theorem and Theorem 5.1.

COROLLARY 6.2. $Q_n(K, L)$ is an ideal of $S_n(K, L)$ with respect to the $**$ product if K and L are finitely generated free cochain complexes.

COROLLARY 6.3. If K and L are finitely generated free cochain complexes, then $S_n(K, L)$ is associative modulo $Q_n(K, L)$, i.e. $x ** (y ** z) \equiv (x ** y) ** z \pmod{Q_n}$.

It can be proved directly that the multiplication defined in $S_n(K, L)$ is unique modulo $Q_n(K, L)$. However if K and L are finitely generated and free, this is an immediate consequence of Theorems 5.1 and 6.1.

Since $Q_n(K, L)$ is an ideal of $S_n(K, L)$ with respect to the $**$ product when K and L are finitely generated and free, $R_n(K, L)$ is a ring. Thus we can state Theorem 5.1 in the stronger form:

THEOREM 6.4. If K and L are finitely generated free cochain complexes, then $\phi_n^*: R_n(K, L) \rightarrow H(K \otimes L, n)$ is a ring isomorphism.

We will now apply the foregoing theory to the example given in the introduction. We let $Z_n[u]$ denote a cyclic group of order n with generator u . Since the Klein bottle K and the union of the real projective plane and a one-sphere ($P_2 \vee S_1$) have only two as torsion number, we need consider only the cohomology rings with integers and integers modulo two for coefficients. The integral cohomology groups and the cohomology groups with integers modulo two are as follows:

$$\begin{aligned} H^0(K) &= H^0(P^2 \vee S_1) = Z[(e)_0], \\ H^1(K) &= H^1(P^2 \vee S_1) = Z[(c)_0], \\ H^2(K) &= H^2(P^2 \vee S_1) = Z_2[(b)_0], \\ H^0(K, 2) &= H^0(P^2 \vee S_1, 2) = Z_2[(e)_2], \\ H^1(K, 2) &= H^1(P^2 \vee S_1, 2) = Z_2[(c)_2] + Z_2[(a)_2], \\ H^2(K, 2) &= H^2(P^2 \vee S_1, 2) = Z_2[(b)_2]. \end{aligned}$$

In each case, these generators of the respective groups are related by the Bockstein and coefficient homomorphisms as follows:

$$\begin{aligned} \Delta_2((a)_2) &= (b)_0, & \Delta_2((e)_2) &= \Delta_2((c)_2) = \Delta_2((b)_2) = 0, \\ h_{2,0}((e)_0) &= (e)_2, & h_{2,0}((c)_0) &= (c)_2 \text{ and } h_{2,0}((b)_0) = (b)_2. \end{aligned}$$

The multiplication table for the cohomology rings of these spaces is determined by giving the products for these generators. With integer coefficients we have the same multiplication table for the both spaces. Thus we have the following table:

	$(e)_0$	$(c)_0$	$(b)_0$
$(e)_0$	$(e)_0$	$(c)_0$	$(b)_0$
$(c)_0$	$(c)_0$	0	0
$(b)_0$	$(b)_0$	0	0

However, with integers modulo two we have different products as indicated in the following tables:

	$(e)_2$	$(c)_2$	$(a)_2$	$(b)_2$
$(e)_2$	$(e)_2$	$(c)_2$	$(a)_2$	$(b)_2$
$(c)_2$	$(c)_2$	0	$(b)_2$	0
$(a)_2$	$(a)_2$	$(b)_2$	$(b)_2$	0
$(b)_2$	$(b)_2$	0	0	0

Klein Bottle K

	$(e)_2$	$(c)_2$	$(a)_2$	$(b)_2$
$(e)_2$	$(e)_2$	$(c)_2$	$(a)_2$	$(b)_2$
$(c)_2$	$(c)_2$	0	0	0
$(a)_2$	$(a)_2$	0	$(b)_2$	0
$(b)_2$	$(b)_2$	0	0	0

$P_2 \vee S_1$

It can be easily seen that the rings determined by the above multiplication tables are not isomorphic. If we compute the cohomology groups of $K \times K$ and $(P_2 \vee S_1) \times (P_2 \vee S_1)$ in dimensions one and three, we find the following generators:

$$(c \otimes e)_0, (e \otimes c)_0 \text{ in dimension one,}$$

$$(c \otimes b)_0, (b \otimes c)_0, \text{ in dimension two}$$

and

$$\left(\frac{1}{2} \delta(a \otimes a) \right)_0 = (b \otimes a - a \otimes b)_0 \text{ in dimension three.}$$

This gives rise to the following multiplication tables:

	$c \otimes e$	$e \otimes c$
$(c \otimes b)_0$	0	0
$(b \otimes c)_0$	0	0
$(b \otimes a - a \otimes b)_0$	$(b \otimes b)_0$	$(b \otimes b)_0$

$K \times K$

	$(c \otimes e)_0$	$(e \otimes c)_0$
$(c \otimes b)_0$	0	0
$(b \otimes c)_0$	0	0
$(b \otimes a - a \otimes b)_0$	0	0

$(P_2 \vee S_1) \times (P_2 \vee S_1)$

This substantiates the assertion made in the introduction.

We now consider a second and more startling example which shows that we actually need to consider not only the abstract ring structures but also the coefficient and Bockstein homomorphisms. Let us take for the spaces, the lens spaces $L(5, 1)$ and $L(5, 2)$ (cf. Seifert and Threlfall [7, p. 210]). We let $L_1 = L(5, 1)$ and $L_2 = L(5, 2)$. Both of these spaces have only five torsion. Hence, we need consider only the cohomology rings with integers and integers modulo five for coefficients along with the corresponding coefficient and Bockstein homomorphisms. It is left to the reader to verify that these spaces have isomorphic cohomology rings with integers and integers mod 5 for coefficients. However, one finds that there is no isomorphism between these rings which commutes with the Bockstein and coefficient homomorphisms. Upon computing the integral cohomology rings of the spaces $L_1 \times L_1$ and $L_2 \times L_2$, using the above constructions, one finds that there can exist no isomorphism (ring) between these rings.

CHAPTER IV. THE COHOMOLOGY SPECTRUM OF $K \otimes L$

7. The Bockstein and coefficient homomorphisms in $S_n(K, L)$. Thus far we have constructed, for each non-negative integer n , a ring $S_n(K, L)$. In this section homomorphisms are defined within this collection of rings which will correspond to the Bockstein and coefficient homomorphisms. These new homomorphisms will be denoted by the same symbols.

For m and n positive integers we define $h_{m,n}: S_n(K, L) \rightarrow S_m(K, L)$ by the formula

$$(7.1) \quad \begin{aligned} h_{m,n}(x, 0) &= \frac{[m, n]}{[m, i]} (h'_{c,i}(x), 0), \\ h_{m,n}(0, x) &= \frac{m}{(m, n)} (\alpha h'_{c,i} D_i(x), \beta h'_{c,i}(x)) \end{aligned}$$

where x is in $H(K, i) \otimes H(L, i)$, $i | n$, $c = (m, i)$ and the integers α and β satisfy $\alpha m + \beta i = c$. We also define $h_{m,0}: S_0(K, L) \rightarrow S_m(K, L)$ by the formula

$$(7.2) \quad \begin{aligned} h_{m,0}(x) &= (h'_{m,0}(x), 0) & i = 0, \\ &= (\alpha h'_{c,i} D_i(x), \beta h'_{c,i}(x)) & i > 0 \end{aligned}$$

for x in $H(K, i) \otimes H(L, i)$, $c = (m, i)$ where the integers α and β satisfy $\alpha m + \beta i = c$.

The Bockstein homomorphism $\Delta_n: S_n(K, L) \rightarrow S_0(K, L)$, $n > 0$, is defined by the formula

$$(7.3) \quad \Delta_n(x, y) = x$$

for (x, y) in $S_n(K, L)$. We also define an involution ω_n of $S_n(K, L)$ for $n > 0$ by the formula

$$\omega_n(x, y) = (\omega'(x), -\omega'(y)) \quad n > 0.$$

An involution ω_0 of $S_0(K, L)$ is defined for x in $H(K, i) \otimes H(L, i)$ by the formula

$$\begin{aligned} \omega_0(x) &= -\omega'(x) & i > 0, \\ &= \omega'(x) & i = 0. \end{aligned}$$

It can be shown directly that $h_{m,n}(Q_n(K, L)) \subset Q_m(K, L)$ and $\Delta_n(Q_n(K, L)) \subset Q_0(K, L)$. However, such a direct proof is a long and tedious computation. The two inclusions are immediate corollaries of the following theorem when K and L are finitely generated and free.

THEOREM 7.1. *The homomorphisms $h_{m,n}$, Δ_n and ω_n defined above commute with ϕ_n . Thus we have the following formulae:*

$$(7.6) \quad \phi_n h_{n,m} = h_{n,m} \phi_m \quad n > 0, m \geq 0,$$

$$(7.7) \quad \phi_0 \Delta_m = \Delta_m \phi_m \quad m > 0,$$

$$(7.8) \quad \phi_n \omega_n = \omega_n \phi_n \quad n \geq 0.$$

The proofs of these formulae are easy verifications and are left to the reader.

As an immediate consequence we have

COROLLARY 7.2. *If K and L are finitely generated free cochain complexes, then*

$$(i) \quad h_{m,n}(Q_n(K, L)) \subset Q_m(K, L), \quad m > 0, n \geq 0;$$

$$(ii) \quad \Delta_n(Q_n(K, L)) \subset Q_0(K, L), \quad n > 0;$$

$$(iii) \quad \omega_n(Q_n(K, L)) \subset Q_n(K, L).$$

Hence these homomorphisms induce homomorphisms $h_{m,n}^*$, Δ_n^* and ω_n^* on the quotient rings $R_n(K, L)$, $n \geq 0$, and these induced homomorphisms satisfy formulae (7.6), (7.7) and (7.8). Thus, since each ϕ_n induces an isomorphism between $R_n(K, L)$ and $H(K \otimes L, n)$, the induced coefficient and Bockstein homomorphisms correspond to the coefficient and Bockstein homomorphisms in the cohomology spectrum of $K \otimes L$.

8. Homomorphisms induced by allowable cochain maps. In this section, the obvious definitions of homomorphisms induced by allowable cochain maps on the rings $S_n(K, L)$ are given. Let K, K', L and L' be cochain complexes, and let $f: K \rightarrow K'$ and $g: L \rightarrow L'$ be allowable cochain homomorphisms. Then $\langle f, g \rangle_0: S_0(K, L) \rightarrow S_0(K', L')$ is defined by the formula

$$(8.1) \quad \langle f, g \rangle_0 | H(K, n) \otimes H(L, n) = f_n \otimes g_n$$

where $f_n: H(K, n) \rightarrow H(K', n)$ and $g_n: H(L, n) \rightarrow H(L', n)$ are the homomorphisms induced by f and g respectively. Let $\langle f, g \rangle'_n = \langle f, g \rangle_0 | P_n(K, L)$. Then $\langle f, g \rangle_n: S_n(K, L) \rightarrow S_n(K', L')$ is defined by the formula

$$(8.2) \quad \langle f, g \rangle_n = (\langle f, g \rangle'_n, \langle f, g \rangle'_n).$$

Since f and g are allowable cochain homomorphisms, f_n and g_n commute with the corresponding Bockstein and coefficient homomorphisms (formula (2.9)). Thus $\langle f, g \rangle_n$, $n \geq 0$ maps the generators of $Q_n(K, L)$ into the corresponding generators of $Q_n(K', L')$ and therefore $\langle f, g \rangle_n$ induces a homomorphism $\langle f, g \rangle_n^*: R_n(K, L) \rightarrow R_n(K', L')$.

The homomorphisms $\langle f, g \rangle_n$ and $\langle f, g \rangle_n^*$, $n \geq 0$ satisfy the following functorial properties:

LEMMA 8.1. (a) *If $f: K \rightarrow K$ and $g: L \rightarrow L$ are identity homomorphisms, then $\langle f, g \rangle_n: S_n(K, L) \rightarrow S_n(K, L)$, $n \geq 0$, and $\langle f, g \rangle_n^*: R_n(K, L) \rightarrow R_n(K, L)$ are identity homomorphisms.*

(b) If $f: K \rightarrow K'$, $f': K' \rightarrow K''$, $g: L \rightarrow L'$ and $g': L' \rightarrow L''$ are allowable homomorphisms, then

$$\langle f'f, g'g \rangle_n = \langle f', g' \rangle_n \langle f, g \rangle_n: S_n(K, L) \otimes S_n(K'', L''),$$

and

$$\langle f'f, g'g \rangle_n^* = \langle f', g' \rangle_n^* \langle f, g \rangle_n^*: R_n(K, L) \rightarrow R_n(K'', L'').$$

These functorial properties follow from the corresponding properties for tensor products, direct sums and homology, the details of the verification being left to the reader.

The following theorem shows that ϕ_n , $n \geq 0$, is a natural homomorphism with respect to induced maps.

THEOREM 8.2. *If $f: K \rightarrow K'$ and $g: L \rightarrow L'$ are allowable cochain homomorphisms, then*

$$\phi_n \langle f, g \rangle_n = (f \otimes g)_n \phi_n: S_n(K, L) \rightarrow H(K' \otimes L', n)$$

and

$$\phi_n^* \langle f, g \rangle_n^* = (f \otimes g)_n \phi_n^*: R_n(K, L) \rightarrow H(K' \otimes L', n).$$

This follows from the definitions, formula (2.9), and the fact that

$$\alpha_i(f_i \otimes g_i) = (f \otimes g)_i \alpha_i.$$

THEOREM 8.3. *Let $f: K \rightarrow K'$ and $g: L \rightarrow L'$ be allowable cochain homomorphisms. Then $\langle f, g \rangle_n: S_n(K, L) \rightarrow S_n(K', L')$ preserves the $*$ $*$ products, i.e. $\langle f, g \rangle_n(x * y) = \langle f, g \rangle_n(x) * \langle f, g \rangle_n(y)$ for x and y in $S_n(K, L)$.*

Proof. It suffices to prove this for the case listed in the definitions of the $*$ $*$ products, e.g. formulae (4.12) and (6.7). The following formulae are used in this verification, where $x \in H(K, i) \otimes H(L, i)$, $y \in H(K, j) \otimes H(L, j)$ and u and v are in $S_0(K, L)$.

$$(8.3) \quad (f_i \otimes g_i) h'_{i,i} = h'_{i,i} (f_i \otimes g_i),$$

$$(8.4) \quad \langle f, g \rangle_0(u \cdot v) = \langle f, g \rangle_0(u) \cdot \langle f, g \rangle_0(v),$$

$$(8.5) \quad D_i(f_i \otimes g_i) = (f_i \otimes g_i) D_i,$$

$$(8.6) \quad \langle f, g \rangle_0(D_i(x) \cdot y) = D_i(\langle f, g \rangle_0(x)) \cdot \langle f, g \rangle_0(y),$$

$$(8.7) \quad \langle f, g \rangle_0(x \cdot D_j(y)) = \langle f, g \rangle_0(x) \cdot D_j \langle f, g \rangle_0(y),$$

$$(8.8) \quad \langle f, g \rangle'_n = \langle f, g \rangle_0 | P_n(K, L).$$

THEOREM 8.4. *If $f: K \rightarrow K'$ and $g: L \rightarrow L'$ are allowable cochain homomorphisms, then*

$$(8.9) \quad \langle f, g \rangle_n h_{n,m} = h_{n,m} \langle f, g \rangle_m: S_m(K, L) \rightarrow S_n(K', L'),$$

$$(8.10) \quad \langle f, g \rangle_0 \Delta_n = \Delta_n \langle f, g \rangle_n : S_n(K, L) \rightarrow S_0(K', L').$$

These formulae are consequences of the definitions. Their proofs are left to the reader.

Formulae (8.9) and (8.10) also hold for the homomorphisms $\langle f, g \rangle_n^*$, $h_{n,m}^*$ and Δ_n^* induced on the quotient rings $R_n(K, L)$.

The results of this section can be interpreted using the language of the theory of functors. (cf. Eilenberg and Steenrod [5, p. 111]).

For each integer $n \geq 0$, S_n is a function which assigns to each pair (K, L) of cochain complexes a ring $S_n(K, L)$ and to each pair of allowable cochain homomorphisms $f: K \rightarrow K'$ and $g: L \rightarrow L'$, a ring homomorphism $\langle f, g \rangle_n : S_n(K, L) \rightarrow S_n(K', L')$. Then Lemma 8.1 states that S_n is a covariant functor on pairs of cochain complexes.

For each $n \geq 0$, R_n is a function which assigns, to each pair (K, L) of finitely generated free cochain complexes, a ring $R_n(K, L)$, and to each pair of allowable cochain maps $f: K \rightarrow K'$ and $g: L \rightarrow L'$, a ring homomorphism $\langle f, g \rangle_n^* : R_n(K, L) \rightarrow R_n(K', L')$. From Lemma 8.1, we see that R_n is a covariant functor defined on pairs of finitely generated free cochain complexes.

9. Extension to infinite complexes and Čech cohomology. In this section, the functors S_n and R_n will be shown to commute with direct limits. This will then enable us to extend the results of the previous sections to infinite cochain complexes. We first list some known results about direct limits. Their proofs, in a general form, may be found in Cartan and Eilenberg [4].

(1) Direct limits commute with tensor products.

(2) Homology, as well as the coefficient and Bockstein homomorphisms, commute with direct limits.

(3) Direct sums commute with direct limits.

We shall use the notation of Eilenberg and Steenrod [5] for direct systems of groups.

Let the cochain complexes $\mathcal{K}$ and $\mathcal{L}$ be the direct limit of the direct system of cochain complexes $\{K, \pi\}$ and $\{L, \rho\}$ over the directed sets M and N respectively. The set $M \times N$ is directed by defining $(\alpha, \beta) < (\gamma, \delta)$ if $\alpha < \gamma$ and $\beta < \delta$ where α and γ are in M and β and δ are in N . Let $\{S_n(K, L), S_n(\pi, \rho)\}$, $\{Q_n(K, L), Q_n(\pi, \rho)\}$ and $\{R_n(K, L), R_n(\pi, \rho)\}$ be the direct system of groups, subgroups and quotient groups defined as follows:

$$\begin{aligned} S_n(K, L)_{(\alpha, \beta)} &= S_n(K_\alpha, L_\beta); & S_n(\pi, \rho)_{(\alpha, \beta)}^{(\gamma, \delta)} &= S_n(\pi_\alpha^\gamma, \rho_\beta^\delta); \\ Q_n(K, L)_{(\alpha, \beta)} &= Q_n(K_\alpha, L_\beta); & Q_n(\pi, \rho)_{(\alpha, \beta)}^{(\gamma, \delta)} &= S_n(\pi_\alpha^\gamma, \rho_\beta^\delta) \mid Q_n(K_\alpha, L_\beta); \\ R_n(K, L)_{(\alpha, \beta)} &= R_n(K_\alpha, L_\beta); & R_n(\pi, \rho)_{(\alpha, \beta)}^{(\gamma, \delta)} &= R_n(\pi_\alpha^\gamma, \rho_\beta^\delta) \text{ for } (\alpha, \beta) < (\gamma, \delta). \end{aligned}$$

THEOREM 9.1. $S_n(\mathcal{K}, \mathcal{L})$, $Q_n(\mathcal{K}, \mathcal{L})$ and $R_n(\mathcal{K}, \mathcal{L})$ are the direct limits of the direct systems of groups, subgroups and quotient groups $\{S_n(K, L), S_n(\pi, \rho)\}$, $\{Q_n(K, L), Q_n(\pi, \rho)\}$ and $\{R_n(K, L), R_n(\pi, \rho)\}$ respectively.

Proof. That $S_n(\mathcal{K}, \mathcal{L})$ is the direct limit of the direct system of groups $\{S_n(K, L), S_n(\pi, \rho)\}$ follows from the facts that tensor products, homology and direct sums commute with direct limits. To see that $Q_n(\mathcal{K}, \mathcal{L})$ is the direct limit of $\{Q_n(K, L), Q_n(\pi, \rho)\}$, one uses the fact that the homology spectrum commutes with direct limits. Thus the relations which define $Q_n(\mathcal{K}, \mathcal{L})$ can be pulled back to some $Q_n(K_\alpha, L_\beta)$. Since direct limits preserve exact sequences (cf. Eilenberg and Steenrod [5, p. 225]), $R_n(\mathcal{K}, \mathcal{L})$ is the direct limit of $\{R_n(K, L), R_n(\pi, \rho)\}$ because for each α in M and β in N , the following sequence is exact:

$$0 \rightarrow Q_n(K_\alpha, L_\beta) \rightarrow S_n(K_\alpha, L_\beta) \rightarrow R_n(K_\alpha, L_\beta) \rightarrow 0.$$

THEOREM 9.2. *If K and L are torsion free cochain complexes, then for each $i \geq 0$,*

- (a) $\phi_i: S_i(K, L) \rightarrow H(K \otimes L, i)$ is a ring homomorphism onto,
- (b) *kernel* $\phi_i = Q_i(K, L)$.

Proof. In §§4 and 6, we saw that ϕ_i is a ring homomorphism for $i \geq 0$. Hence it suffices to prove that ϕ_i is onto and *kernel* $\phi_i = Q_i(K, L)$. K is the union of its finitely generated allowable subgroups. Also for each pair (K_α, K_β) of finitely generated allowable subgroups, the subgroup generated by $K_\alpha \cup K_\beta$ is finitely generated and allowable. Hence K is the direct limit of its finitely generated allowable subgroups (cf. Eilenberg and Steenrod [5, p. 229]). Similarly L is the direct limit of its finitely generated allowable subgroups.

Since a finitely generated subgroup of a torsion free group is free, the following sequence is exact for any finitely generated allowable subgroups K_α of K and L_β of L :

$$0 \rightarrow Q_i(K_\alpha, L_\beta) \rightarrow S_i(K_\alpha, L_\beta) \rightarrow H(K_\alpha \otimes L_\beta, i) \rightarrow 0$$

where the map $S_i(K_\alpha, L_\beta) \rightarrow H(K_\alpha \otimes L_\beta, i)$ is $\phi_{(\alpha, \beta)}$. Since direct limits preserve exact sequences, we conclude by Theorem 9.1 that the following sequence is exact:

$$0 \rightarrow Q_i(K, L) \rightarrow S_i(K, L) \rightarrow H(K \otimes L, i) \rightarrow 0$$

where the map $S_i(K, L) \rightarrow H(K \otimes L, i)$ is ϕ_i^∞ . Since ϕ_i is natural, it commutes with the induced projections; hence $\phi_i^\infty = \phi_i$. This completes the proof of the theorem.

COROLLARY 9.3. *If K and L are torsion free cochain complexes, then $\phi_i^*: R_i(K, L) \rightarrow H(K \otimes L, i)$ is a ring isomorphism onto.*

Let X be a locally compact space, and let $H(X, i)$ be the Čech cohomology ring of X with compact supports with the integers modulo i as coefficients. It is known (cf. H. Cartan [3] or J. Leray [6]) that a torsion free cochain complex K can be assigned to X so that the cohomology ring $H(K, i)$ is naturally isomorphic to $H(X, i)$. It is also known that if K and L are torsion free cochain

complexes assigned to the locally compact spaces X and Y , as above, then the torsion free cochain complex $K \otimes L$ has the property that the cohomology ring $H(K \otimes L, i)$ is naturally isomorphic to $H(X \times Y, i)$ (for details and a proof see [3, p. 60] or [6, pp. 113–114]). Thus we see that the cohomology spectra of X and Y determine the cohomology rings $H(X \times Y, i)$. This is stated precisely in

COROLLARY 9.4. *If X and Y are locally compact spaces, then $\phi_i^*: R_i(X, Y) \rightarrow H(X \times Y, i)$ is a ring isomorphism onto.*

Proof. Since the spectra of X and Y are isomorphic to the spectra of K and L , where K and L are as above, we have $R_i(X, Y)$ is isomorphic to $R_i(K, L)$. By Corollary 9.3, $R_i(K, L)$ is isomorphic to $H(K \otimes L, i)$ which in turn is isomorphic to $H(X \times Y, i)$.

THEOREM 9.5. *If X and Y are compact spaces, the Čech cohomology spectra of X and Y determine the Čech cohomology rings $H(X \times Y, i)$, $i \geq 0$.*

Proof. The Čech cohomology ring, $H(X \times Y, i)$ is the direct limit of cohomology rings of nerves of finite coverings. One can verify that the set of "product" coverings $\{U_i \times V_i\}$ with $\{U_i\}$ a finite open covering of X and $\{V_i\}$ a finite open covering of Y is cofinal in the set of all coverings of $X \times Y$. Let $N(U)$ denote the nerve of a covering U . It can be verified that the simplicial product ([5, pp. 66–70]) $N(U) \Delta N(V) = N(U \times V)$ where $U \times V$ is the product covering. Since $|N(U) \Delta N(V)|$ is homotopically equivalent to $|N(U) \times N(V)|$, it follows that $H(N(U) \times N(V))$ is naturally isomorphic to $H(N(U \times V))$. Hence

$$\phi_i^\infty = \lim_{(U,V)} \phi_i^{(U,V)}$$

is an isomorphism of $R_i(X, Y)$ onto $H(X \times Y, i)$ by Theorem 9.1.

Appendix. We shall prove Theorems 3.1 and 5.1 for the case where K and L are elementary complexes with two free generators. Let K have free generators a and b in dimensions p and $p+1$ respectively with $\delta a = tb$, $\delta b = 0$, and let L have free generators a' and b' in dimensions q and $q+1$ respectively with $\delta a' = sb'$, $\delta b' = 0$. For easy reference, we list the cohomology groups of K , L and $K \otimes L$ with the integers mod n for coefficients, $n \geq 0$. $Z_n[u]$ will denote a cyclic group of order n with u as a generator, $c = (n, t, s)$ and $d = (t, s)$.

$$\begin{aligned} H^p(K, n) &= Z_{(t,n)} \left[\left(\frac{n}{(n, t)} a \right)_n \right], & n > 0, \\ H^{p+1}(K, n) &= Z_{(t,n)} [b_n], & n \geq 0, \\ H^q(L, n) &= Z_{(s,n)} \left[\left(\frac{n}{(n, s)} a' \right)_n \right], & n > 0, \end{aligned}$$

$$\begin{aligned}
H^{q+1}(L, n) &= Z_{(s, n)}[b'_n], & n \geq 0, \\
H^{p+q}(K \otimes L, n) &= Z_c \left[\left(\frac{n}{c} a \otimes a' \right)_n \right], & n > 0, \\
H^{p+q+1}(K \otimes L) &= Z_d \left[\left(\frac{1}{d} \delta(a \otimes a') \right)_0 \right], \\
H^{p+q+2}(K \otimes L, n) &= Z_c [(b \otimes b')_n], & n \geq 0, \\
H^{p+q+1}(K \otimes L, n) &= Z_c + Z_c, & n > 0.
\end{aligned}$$

It is convenient to describe this last group as the abelian group generated by

$$E = \left(\frac{1}{d} \delta(a \otimes a') \right)_n, \quad G = (-1)^p \left(\frac{n}{(n, t)} a \otimes b' \right)_n$$

and

$$F = \left(\frac{n}{(n, s)} b \otimes a' \right)_n$$

subject to the relations:

$$\begin{aligned}
cE &= cF = cG, \\
\frac{nc}{(n, t)(n, s)} E &= \frac{tc}{(n, t)d} F + \frac{sc}{(n, s)d} G.
\end{aligned}$$

All the other cohomology groups of K , L and $K \otimes L$ are zero. The proof that the group generated by E , F and G subject to the above relations is the direct sum of two cyclic groups of order c is left to the reader. Note that the integers $nc/[(n, t)(n, s)]$, $tc/[(n, t)d]$ and $sc/[(n, s)d]$ are relatively prime in pairs.

To prove that ϕ_n is onto, it suffices to exhibit a pre-image for each generator. Thus we have

$$\begin{aligned}
\phi_n(a_c \otimes a'_c, 0) &= \left(\frac{n}{c} a \otimes a' \right)_n, & n > 0, \\
\phi_0(a_d \otimes a'_d) &= \left(\frac{1}{d} \delta(a \otimes a') \right)_0, \\
\phi_n(b_n \otimes b'_n, 0) &= (b \otimes b')_n, & n > 0, \\
\phi_0(b'_0 \otimes b_0) &= (b \otimes b')_0, \\
\phi_n(b_{(n, s)} \otimes a'_{(n, s)}, 0) &= F, \\
\phi_n((-1)^p a_{(n, t)} \otimes b'_{(n, t)}, 0) &= G, \\
\phi_n(\alpha h'_{c, d} D_d(a_d \otimes a'_d), \beta a_c \otimes a'_c) &= E,
\end{aligned}$$

where α and β are integers such that $\alpha n + \beta d = c$. The verification of the above formulae is left to the reader.

It is easy to see that $Q_n(K, L) \subset \text{kernel } \phi_n$, $n \geq 0$, because ϕ_n maps each generator of $Q_n(K, L)$ into zero. Thus to prove (b) $\text{kernel } \phi_n = Q_n(K, L)$, it suffices to prove (b') $\text{kernel } \phi_n \subset Q_n(K, L)$, $n \geq 0$.

In proving (b'), it will be convenient to introduce a graded structure in $S_n(K, L)$, $n \geq 0$. This is done for $S_0(K, L)$ by defining

$$S_0^r(K, L) = \sum_{p+q=r} H^p(K) \otimes H^q(L) + \sum_{p+q=r-1} \sum_{n>0} H^p(K, n) \otimes H^q(L, n).$$

For $S_n(K, L)$, we define

$$P_n^r(K, L) = \sum_{p+q=r} \sum_{i|n} H^p(K, i) \otimes H^q(L, i)$$

and

$$S_n^r(K, L) = P_n^r(K, L) \times P_n^{r-1}(K, L).$$

It is easily seen that $\phi_n(S_n^r(K, L)) \subset H^r(K \otimes L, n)$, $n \geq 0$. Thus to prove that $\text{kernel } \phi_n \subset Q_n(K, L)$, $n \geq 0$, it suffices to consider only homogeneous elements of $S_n(K, L)$.

An equivalence relation $\sim$ is introduced in $P_n(K, L)$ as follows: $u \sim v$ if $u - v$ is a linear combination of elements of the form $h_{i,j}(x) \otimes y - x \otimes h_{j,i}(y)$ where both i and j divide n . That $\sim$ is an equivalence relation follows immediately.

We now list some relations which are useful in the proof of b' . We assume that i divides n and $c = (n, t, s)$.

$$(A.1) \quad h_{i,t}(x) \otimes h_{i,s}(y) \sim \frac{i(n, t, s)}{(i, t)(i, s)} h_{c,t}(x) \otimes h_{c,s}(y),$$

$$(A.2) \quad \sum_{i|n} A_i h_{i,t}(u) \otimes h_{i,s}(v) \sim \left(\sum_{i|n} A_i \frac{i(n, t, s)}{(i, t)(i, s)} \right) (h_{c,t}(u) \otimes h_{c,s}(v))$$

where each A_i is an integer.

$$(A.3) \quad h_{i,n}(x) \otimes h_{i,t}(y) \sim \frac{(n, t)}{(i, t)} x \otimes h_{n,t}(y),$$

$$(A.4) \quad h_{i,t}(x) \otimes h_{i,n}(y) \sim \frac{(n, t)}{(i, t)} h_{n,t}(x) \otimes y,$$

$$(A.5) \quad D_i(u) \sim D_n h'_{n,i}(u) \text{ for } u \text{ in } H(K, i) \otimes H(L, i).$$

Formulae (A.1), (A.3), (A.4) and (A.5) follow from formula (2.1), and the definition of $\sim$, while (A.2) follows from (A.1).

An equivalence relation is introduced in $S_n(K, L) = P_n(K, L) \times P_n(K, L)$ as follows: For u and v in $S_n(K, L)$, $u \equiv v \pmod{Q_n}$ if $u - v$ is in $Q_n(K, L)$. It is immediate that this is an equivalence relation.

For x and y in $P_n(K, L)$ the following three statements are equivalent:

- (i) $x \sim y$,
- (ii) $(0, x) \equiv (0, y) \pmod{Q_n}$,
- (iii) $(x, 0) \equiv (y, 0) \pmod{Q_n}$.

This follows from the definitions of $Q_n(K, L)$ and the equivalence relations defined above.

The following equivalences modulo Q_n are readily verified. A proof of (A.7) is given below. In each case i divides n .

$$(A.6) \quad (0, h_{i,0}(x) \otimes h_{i,0}(y)) \equiv 0 \pmod{Q_n},$$

$$(A.7) \quad (D_i(u), 0) \equiv \left(0, \frac{n}{i} u\right) \pmod{Q_n},$$

$$(A.8) \quad (0, D_i(u)) \equiv 0 \pmod{Q_n}$$

for u in $H(K, i) \otimes H(L, i)$, x in $H(K)$ and y in $H(L)$.

Proof of (A. 7). It suffices to prove this for $u = x \otimes y$ in $H(K, i) \otimes H(L, i)$. $(n/i)x \otimes y = h_{i,n}h_{n,i}(x) \otimes y \sim h_{n,i}(x) \otimes h_{n,i}(y)$ by formula (2.1) and the definition of $\sim$. Hence $(0, (n/i)x \otimes y) \equiv (0, h_{n,i}(x) \otimes h'_{n,i}(y))$. By formula (A.5) and the relations defining $Q_n(K, L)$ this is $\equiv (D_n h'_{n,i}(x \otimes y), 0) \equiv (D_i(x \otimes y), 0)$. The next two formulae are proved in a similar fashion.

$$(A.9) \quad \frac{d}{c} (\alpha h'_{c,d} D_d(u), \beta h'_{c,d} u) \equiv (0, h'_{c,d}(u)) \pmod{Q_n},$$

$$(A.10) \quad \frac{nc}{(n, t)(n, s)} e \equiv \frac{tc}{(n, t)d} f + \frac{sc}{(n, s)d} g \pmod{Q_n}$$

where $d = (t, s)$, $c = (n, t, s)$, $u \in H(K, d) \otimes H(L, d)$, α and β are integers such that $\alpha n + \beta d = c$, $e = (\alpha h'_{c,d} D_d(h_{d,t}(x_t) \otimes h_{d,s}(y_s)), \beta h_{c,t}(x_t) \otimes h_{c,s}(y_s))$, $f = (h_{n,0} \Delta_t(x_t) \otimes h_{n,s}(y_s), 0)$ and $g = (h_{n,t} \omega(x_t) \otimes h_{n,0} \Delta_s(y_s), 0)$, $x_t \in H(K, t)$ and $y_s \in H(L, s)$. The proof that kernel $\phi_n \subset Q_n(K, L)$ can be broken down into the following cases for $n > 0$.

$$\text{Case (1)} \quad x \in S_n^{p+q+3}(K, L).$$

$$\text{Case (2)} \quad x \in S_n^{p+q+2}(K, L).$$

$$\text{Case (3)} \quad x \in S_n^{p+q+1}(K, L).$$

$$\text{Case (4)} \quad x \in S_n^{p+q}(K, L).$$

We will only give proofs for Case (1), $n > 0$, and Case (3), $n > 0$ because the proofs for the other cases are similar.

Proof of Case 1, $n > 0$. Let $x \in S_n^{p+q+3}(K, L)$, $n > 0$. Then $\phi_n(x) = 0$ because

$H^{p+q+3}(K \otimes L, n) = 0$. But $x = \sum_{i|n} A_i(0, h_{i,0}(b_0) \otimes h_{i,0}(b'_0))$ which by formula (A.6) is $\equiv 0 \pmod{Q_n}$. Thus x is in $Q_n(K, L)$.

Proof of Case 3, $n > 0$. Suppose $x \in S_n^{p+q+1}(K, L)$. Then by formulae (A.3), (A.4), (A.9), and (A.2), $x \equiv Ae + Bf + Cg \pmod{Q_n}$ where A, B , and C are integers, $c = (n, t, s)$, $d = (t, s)$, α and β are integers such that $\alpha n + \beta d = c$,

$$\begin{aligned} e &= (\alpha h'_{c,d} D_d(a_d \otimes a'_d), \beta h_{c,t}(a_t) \otimes h_{c,s}(a'_s)), \\ f &= (b_n \otimes h_{n,s}(a'_s), 0) \end{aligned}$$

and

$$g = (h_{n,t}(\omega(a_t)) \otimes b'_n, 0).$$

Since $\phi_n(e) = E$, $\phi_n(f) = F$ and $\phi_n(g) = G$, $\phi_n(x) = 0$ implies $AE + BF + CG = 0$. Thus

$$\begin{aligned} A &\equiv m \frac{nc}{(n, t)(n, s)} \pmod{c}, \\ B &\equiv -m \frac{tc}{(n, t)d} \pmod{c}, \\ C &\equiv -m \frac{sc}{(n, s)d} \pmod{c} \end{aligned}$$

for some integer m . Since $ce = cf = cg$, this implies by formula (A.10) that $x \equiv 0 \pmod{Q_n}$.

One proceeds in a similar fashion in the (simpler) case $n = 0$.

BIBLIOGRAPHY

1. M. Bockstein *Universal systems of Δ -homology rings* C. R. (Doklady) Acad. Sci. URSS. vol. 37 (1942) pp. 243–245.
2. ———, *Homological invariants of the topological product of two spaces*, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 339–342.
3. H. Cartan, *Seminaire de Topologie Algebrique* ENS, III, 1950–1951.
4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, 1956.
5. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.
6. J. Leray, *L'anneau spectra et l'anneau filtre d'homologie d'une espace localement compact et d'une application continue*, J. Math. Pures Appl. vol. 29 (1950) pp. 1–139.
7. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934.
8. J. H. C. Whitehead, *On simply-connected, four-dimensional polyhedra*, Comment. Math. Helv. vol. 22 (1949) pp. 48–90.

BROWN UNIVERSITY
PROVIDENCE, R.I.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH.